

Umbral Deformations on Discrete Spacetime

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Abstract

Given a minimum measurable length underlying spacetime, the latter may be effectively regarded as discrete, at scales of order the Planck length. A systematic discretization of continuum physics may be effected most efficiently through the umbral deformation. General functionals yielding such deformations at the level of solutions are furnished and illustrated, and broad features of discrete oscillations and wave propagation are outlined.

1 Introduction

A generic theme of robust arguments [1] has established an expectation of a fundamental minimum measurable length in nature, of the order of $l_{Planck} \equiv \sqrt{\hbar G_N/c^3} \sim 1.6 \cdot 10^{-35}m$, the corresponding time for which is l_{Planck}/c . The essence of such arguments is the following (in geometrical Planck units, in which \hbar , c , and $m_{Planck} \equiv \sqrt{\hbar c/G_N}$ are chosen to be 1). In a system or process characterized by energy E , no lengths smaller than L can be accessed or measured, where L is the *larger* of either the Schwarzschild horizon radius of the system, $\sim E$, or, for energies smaller than the Planck mass, the Compton wavelength of the aggregate process, $\sim 1/E$. Since the minimum of $\max(E, 1/E)$ lies at the Planck mass, $E \sim 1$, the smallest measurable distance is widely recognized to amount to l_{Planck} .

Thus, continuum laws in nature are expected to be deformed, in principle, by modifications of $O(l_{Planck})$. Remarkably, even as something like a fundamental spacetime lattice of spacing $a = O(l_{Planck})$ is thus likely to underlie conventional physics, continuous symmetries, such as Galilei or Lorentz invariance, can actually survive unbroken such a deformation into discreteness, in a nonlocal, *umbral realization* [2].

Umbral calculus, pioneered by Rota and associates in combinatorics contexts [3, 4], specifies, in principle, how functions of discrete variables representing observables get to “shadow” their continuum limit ($a \rightarrow 0$) *systematically*, and effectively preserve Leibniz’s chain rule and the Lie Algebras of the difference operators which shadow (deform) the standard differential operators of continuum physics. (For a review useful to physics, see [5].)

Nevertheless, while the continuous symmetries and Lie Algebras of umbrally deformed systems might remain identical to their continuum limit, the functions of observables themselves are modified, in general, often drastically so. Usually, the controlling continuum differential equations of physics

are discretized [6]. Then, such discrete difference equations are solved to yield umbral deformations of the continuum solutions. However, some complication may be bypassed by umbrally deforming the continuum solutions directly. Still, as illustrated below on the simplest case of oscillations and wave propagation, the modifications may become technically cumbersome in extracting the actual physics involved.

In this brief note, an explicit deforming functional more tractable than the abstract umbral deformation definition is articulated, (17,34), which is suited to transforming continuum solutions. It is an integral functional shadowing the Fourier representation. It should be of utility in inferring wave disturbance propagation in discrete spacetime lattices, by evaluating the umbral transforms of standard continuum physics quantities. Can generic features of discrete spacetime impact wave propagation at cosmic distances? We first illustrate the umbral language on harmonic oscillations discretely deformed through asymmetric finite differences, Δ_+ , for simplicity, so the reader could quickly confirm all statements by standard numerical analysis textbook methods (Section 2). We then point out how the umbral transform kernel technique proposed may bypass conventional umbral calculus complications for more realistic symmetric differences Δ_s , for which basic polynomial sets and umbral manipulations are less straightforward (Section 3). We conclude by discussing generic features of wave propagation on such lattices at cosmic distances, and the logical possibilities afforded by the index of refraction modifications at $O(l_{Planck})$; we finally propose conceivable solitonic applications (Section 4).

2 Overview of the Umbral Correspondence

For simplicity, consider discrete time, $t = 0, a, 2a, \dots, na, \dots$. Without loss of generality, broadly following the summary review of [5], consider first the Δ_+ discretization (umbral deformation) of ∂_t ,

$$\Delta x(t) \equiv \frac{x(t+a) - x(t)}{a}, \quad (1)$$

and whence of the elementary oscillation equation, $\ddot{x}(t) = -x(t)$, namely

$$\Delta^2 x(t) = \frac{x(t+2a) - 2x(t+a) + x(t)}{a^2} = -x(t). \quad (2)$$

and investigate the periodicity of its solutions. Of course, this can be easily solved directly by the textbook Fourier-component Ansatz $x(t) \propto r^t$, [7], to yield $(1 \pm a)^{t/a}$. To illustrate the powerful systematics of umbral calculus [5], however, we produce and study the solution in that framework, instead.

The umbral framework considers associative chains of operators, generalizing ordinary continuum functions by ultimately acting on a translationally-invariant “Fock vacuum”, 1, after manipulations to move shift operators to the right and have them absorbed in that vacuum. Using the standard Lagrange-Boole shift generator

$$T \equiv e^{a\partial_t}, \quad \text{so that} \quad T f(t) \cdot 1 = f(t+a) T \cdot 1 = f(t+a), \quad (3)$$

the umbral deformation is

$$\partial_t \quad \mapsto \quad \Delta \equiv \frac{T - 1}{a}, \quad (4)$$

$$t \mapsto tT^{-1}, \quad (5)$$

$$t^n \mapsto (tT^{-1})^n = t(t-a)(t-2a)\dots(t-(n-1)a)T^{-n} \equiv [t]^n T^{-n}, \quad (6)$$

so that $[t]^0 = 1$, and, for $n > 0$, $[0]^n = 0$. These are called basic polynomials [3, 5, 6].

A linear combination of monomials (a power series representation of a function) will thus transform umbrally to the same linear combination of basic polynomials, with the same series coefficients, $f(t) \mapsto f(tT^{-1})$. All observables in the discretized world are thus such deformation maps of the continuum observables, and evaluation of their direct functional form is in order. Below, we will be concluding the correspondence by casually eliminating translation operators at the very end, through operating on 1, so that $f(tT^{-1}) \cdot 1$.

The umbral deformation relies on the respective umbral entities obeying operator combinatorics identical to their continuum limit ($a \rightarrow 0$), by virtue of obeying the *same Heisenberg commutation relation*,

$$[\partial_t, t] = \mathbb{1} = [\Delta, tT^{-1}]. \quad (7)$$

(Formally, the umbral deformation reflects (unitary) equivalences of the unitary irreducible representation of the Heisenberg-Weyl group, provided for by the Stone-von Neumann theorem. Here, these equivalences reflect the alternate consistent realizations of all continuum physics structures through systematic maps such as the one illustrated here.)

Thus, e.g., by shift invariance, $T\Delta T^{-1} = \Delta$,

$$[\partial_t, t^n] = nt^{n-1} \mapsto [\Delta, [t]^n T^{-n}] = n[t]^{n-1} T^{1-n}, \quad (8)$$

so that, ultimately, $\Delta[t]^n = n[t]^{n-1}$.

Likewise [3, 8],

$$[t]^n T^{-n} [t]^m T^{-m} \equiv [t]^n \ast [t]^m T^{-n-m} = [t]^{n+m} T^{-(n+m)}, \quad (9)$$

and so forth. The right member of the equality is the implicit definition of the product [8] defined by dotting on 1, $[t]^n \ast [t]^m \equiv [t]^{n+m}$.

For commutators of associative operators, the umbrally deformed Leibniz rule holds [8, 2],

$$[\Delta, f(tT^{-1})g(tT^{-1})] = [\Delta, f(tT^{-1})]g(tT^{-1}) + f(tT^{-1})[\Delta, g(tT^{-1})], \quad (10)$$

ultimately to be dotted onto 1.

Now note that, in this case, the basic polynomials $[t]^n$ are just scaled falling factorials $a^n(t)_n$,

$$[t]^n = a^n \frac{(t/a)!}{(t/a - n)!}, \quad (11)$$

so that $[-t]^n = (-)^n [t+a(n-1)]^n$. (Furthermore, $[an]^n = a^n n!$; for $0 \leq m \leq n$, $[t]^m [t-am]^{n-m} = [t]^n$; and for integers $0 \leq m < n$, $[am]^n = 0$. Thus, $\Delta^m [t]^n = [an]^m [t]^{n-m}/a^m$.)

The standard umbral exponential is then natural to define as, [3, 8, 9],

$$E(\lambda t, a) \equiv e^{\lambda[t]} \equiv e^{\lambda t T^{-1}} \cdot 1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [t]^n = \sum_{n=0}^{\infty} (\lambda a)^n \binom{t/a}{n} = (1 + \lambda a)^{t/a}, \quad (12)$$

the compound interest formula, with the proper continuum limit ($a \rightarrow 0$). Evidently, since $\Delta \cdot 1 = 0$,

$$\Delta e^{\lambda[t]} = \lambda e^{\lambda[t]}, \quad (13)$$

and, as already indicated, one could have solved this equation directly to produce the above $E(\lambda t, a)$ ¹.

Serviceably, the umbral exponential E happens to be an ordinary exponential,

$$e^{\lambda[t]} = e^{\frac{\ln(1+\lambda a)}{a}t}. \quad (14)$$

The umbral exponential actually serves as the generating function of the umbral basic polynomials,

$$\left. \frac{\partial^n}{\partial \lambda^n} (1 + \lambda a)^{t/a} \right|_{\lambda=0} = [t]^n. \quad (15)$$

Conversely, then, this construction may be reversed, by first solving directly for the umbral eigenfunction of Δ , and effectively defining the umbral basic polynomials through the above parametric derivatives, in situations where these might be more involved, as in the next section.

By linearity, the umbral deformation of a power series representation of a function formally evaluates to

$$f(t) \mapsto F(t) \equiv f(tT^{-1}) \cdot 1 = f\left(\frac{\partial}{\partial \lambda}\right) (1 + \lambda a)^{t/a} \Big|_{\lambda=0}, \quad (16)$$

as a consequence. This may not always be easy to evaluate, but, in fact, the same argument may be applied to linear combinations of exponentials, and hence the Fourier representation, instead,

$$F(t) = \int_{-\infty}^{\infty} d\tau f(\tau) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\tau k} (1 + ika)^{t/a} = \left(1 + a \frac{\partial}{\partial \tau}\right)^{t/a} f(\tau) \Big|_{\tau=0}. \quad (17)$$

The rightmost equation follows by converting k into ∂_τ derivatives and integrating by parts away from the resulting delta function. Naturally, it identifies with the above eqn (16) by the (Fourier) identity $f(\partial_x)g(x)|_{x=0} = g(\partial_x)f(x)|_{x=0}$. It is up to the ingenuity of the particular application of such functionals involved to utilize the form whose domain of applicability is best suited for the evaluation sought.

It is also straightforward to check that this umbral transform functional yields

$$\partial_t f \mapsto \Delta F; \quad (18)$$

or to evaluate the umbral transform of the Dirac delta function, which amounts to a cardinal sine, or sampling, function,

$$\delta(t) \mapsto \frac{\sin(\frac{\pi}{2}(1 + t/a))}{(\pi(a + t))}; \quad (19)$$

¹ There is an infinity of nonumbral extensions of this solution, however: multiplying this umbral exponential by an arbitrary periodic function $g(t + a) = g(t)$ will not be visible to Δ , and thus will also yield an eigenfunction of Δ . Often, as below, such extra solutions have a nonumbral, vanishing, continuum limit, or an ill-defined one.

or to evaluate

$$f = \frac{1}{(1-t)} \mapsto F = e^{1/a} a^{t/a} \Gamma(t/a + 1, 1/a), \quad (20)$$

an incomplete Gamma function, and so on. (For example, discrete integration of the above cardinal sine function leads to the umbral transform of the step function, a hypergeometric function ${}_2F_1$ of imaginary argument.) In practical applications, evaluation of umbral transforms of arbitrary functions of observables may well be more direct, at the level of solutions, through this deforming functional, eqn (17).

For example, one may evaluate in this way the umbral correspondents of trigonometric functions,

$$\sin[t] \equiv \frac{e^{i[t]} - e^{-i[t]}}{2i}, \quad \cos[t] \equiv \frac{e^{i[t]} + e^{-i[t]}}{2}, \quad (21)$$

so that

$$\Delta \sin[t] = \cos[t], \quad \Delta \cos[t] = -\sin[t]. \quad (22)$$

Thus, the umbral deformation of phase-space rotations,

$$\dot{x} = p, \quad \dot{p} = -x \quad \mapsto \quad \Delta x(t) = p(t), \quad \Delta p(t) = -x(t), \quad (23)$$

readily yields, by directly deforming continuum solutions, the oscillatory solutions,

$$x(t) = x(0) \cos[t] + p(0) \sin[t], \quad p(t) = p(0) \cos[t] - x(0) \sin[t]. \quad (24)$$

As indicated, the umbral exponential being an ordinary exponential; and, by

$$(1 + ia) = \sqrt{1 + a^2} e^{i \arctan(a)}, \quad (25)$$

these solutions are seen to actually amount to discrete phase-space spirals,

$$x(t) = (1 + a^2)^{\frac{t}{2a}} \left(x(0) \cos(\omega t) + p(0) \sin(\omega t) \right), \quad p(t) = (1 + a^2)^{\frac{t}{2a}} \left(p(0) \cos(\omega t) - x(0) \sin(\omega t) \right), \quad (26)$$

with a frequency *decreased* from the continuum value 1 to

$$\omega = \arctan(a)/a \leq 1, \quad (27)$$

effectively the inverse of the cardinal tangent function.

That is, for $\theta \equiv \arctan(a)$, the spacing of the zeros, period, etc, are scaled up by a factor of

$$\text{tanc}(\theta) \equiv \frac{\tan(\theta)}{\theta} \geq 1. \quad (28)$$

(For complete periodicity on the time lattice, one further needs return to the origin in an integral number of N steps, thus a solution of $N = 2\pi n / \arctan(a)$.) Example: For $a = 1$, the solutions' radius spirals out as $2^{t/2}$, while $\omega = \pi/4$, and the period is $\tau = 8$.

Note that the umbrally conserved quantity is,

$$2\mathcal{E} = x(t) * x(t) + p(t) * p(t) = x(0)^2 + p(0)^2 = (1 + a^2)^{\frac{-t}{a}} \left(x(t)^2 + p(t)^2 \right), \quad (29)$$

($\Delta \mathcal{E} = 0$), with the proper energy as the continuum limit.

3 More Symmetric Cases

Unfortunately, the Δ_+ of eqn (1) is not time-reversal odd, and thus its square is not time-reversal invariant—whence the awkward outspiraling of the solutions of the previous section. (Its time-reversal conjugate, Δ_- , would have in-spiraling solutions.)

Instead, one often chooses half the difference of these two delta operators, i.e., the time-reversal-odd umbral deformation,

$$\partial_t \quad \mapsto \quad \Delta_s \equiv \frac{T - T^{-1}}{2a} . \quad (30)$$

The eigenfunctions of $\Delta_s E = \lambda E$ are now two [10],

$$E_{\pm} = \left(\lambda a \pm \sqrt{1 + (\lambda a)^2} \right)^{t/a} ; \quad (31)$$

one, E_+ , going to the exponential in the continuum limit; but the other, E_- (“nonumbral”), simply oscillating to zero—an oscillation of infinite frequency.

Now, since $[\Delta_s, 2t/(T + T^{-1})] = \mathbb{1}$, the basic polynomials, $[t]_s^n = (t \cdot 2/(T + T^{-1}))^n \cdot 1$, would be harder to evaluate, in principle. Thus, they have been evaluated [6] from $\Delta_s[t]_s^n = n[t]_s^{n-1}$ instead,

$$[t]_s^n = t \prod_{k=1}^{n-1} \left(t + a(n - 2k) \right) . \quad (32)$$

However, according to the general generating function consideration of the previous section, they may alternatively be generated more directly from E_+ ,

$$[t]_s^n = \frac{\partial^n}{\partial \lambda^n} \left(\lambda a + \sqrt{1 + (\lambda a)^2} \right)^{t/a} \Big|_{\lambda=0} . \quad (33)$$

(E.g., the reader may easily check that $[t]_s^3 = (t+a)t(t-a)$, etc.) Since $E_- E_+ = (-)^{t/a}$, E_- generates the reflected basis $[t]_s^n (-)^{t/a+n}$. In general, finding the eigenfunctions of a given difference operator and utilizing them as generating functions of basic polynomial sets may provide shortcuts in effort.

Rather than using umbral deformations of power series representations for functions, however, one may instead infer, *mutatis mutandis* as in (17), umbral transforms of Fourier representations,

$$f(t) \quad \mapsto \quad F_s(t) = \int_{-\infty}^{\infty} d\tau f(\tau) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\tau k} \left(ika + \sqrt{1 - (ka)^2} \right)^{t/a} , \quad (34)$$

to evaluate umbral deformations for general observables, as well as nonumbral ones relying on E_- , with a minus sign in the above kernel of the deforming functional.

Thus, now there are four solutions to the analog of eqn (2),

$$(\Delta_s^2 + 1)x(t) = 0 , \quad (35)$$

namely

$$x(t) = \left(\pm ia \pm \sqrt{1-a^2} \right)^{t/a}. \quad (36)$$

Thus the discrete-time solution set

$$x(t) = (-)^{Nt/a} \left(x(0) \cos(\omega t) + p(0) \sin(\omega t) \right), \quad p(t) = (-)^{Nt/a} \left(p(0) \cos(\omega t) - x(0) \sin(\omega t) \right), \quad (37)$$

maps onto itself under time-reversal, for integer parameter $N = 0, 1$. ($\Delta_s x(0) = (-)^N p(0)$.) This eqn (35) only connects even points on the time lattice among themselves, and odd ones among themselves. Thus, all even points on the time lattice behave the same for even or odd parameter N . However, for the $N = 1$ solutions, the odd time points hop out of phase by π (reflection with respect to the origin in phase space), as they are not dynamically linked to the even points², a phenomenon familiar in lattice gauge theory.

For $N = 0$, the frequency is *increased* over its continuum limit:

$$\omega = \arcsin(a)/a \geq 1. \quad (38)$$

For $N = 1$, the arcsine effectively advances by π and the frequency has an additional component of π/a . Thus, these nonumbral solutions collapse to 0 in the continuum limit.

The conserved energy is more conventional,

$$2\mathcal{E}_s = x(t)^2 + p(t)^2. \quad (39)$$

This reversal-odd difference operator (30) is the one to be considered in wave propagation in the next section, to avoid presumably unphysical exponential amplitude modulations, growths or dwindlings, peculiar to the asymmetric derivative seen in the previous section.

4 Wave Propagation Outlines

Given the features of discrete oscillations outlined, simple plane waves in a positive or negative direction x would obey an equation of the type [9, 10],

$$(\Delta_x^2 - \Delta_t^2) F_s = 0, \quad (40)$$

with the symmetric difference operators of the type (30) on a time lattice with spacing a , and an x -lattice of spacing b , respectively, not necessarily such that $b = a$ in some spacetime regions.

For generic frequency, wavenumber and velocity, the basic right-moving waves $e^{i(\omega t - kx)}$ have phase velocity

$$v(\omega, k) = \frac{\omega}{k} \frac{a \arcsin(b)}{b \arcsin(a)}, \quad (41)$$

²Actually, as above, if $f(t)$ is a solution of (35), $g(t)f(t)$ will also be a solution for arbitrary periodic $g(t+2a) = g(t)$. Thus, even though $(-)^{t/a}$ is one such possible $g(t)$, there are even *more* solutions with arbitrarily mismatched moduli (phase-space radii) and phases between the odd and even sublattices—only their frequencies of rotation need be the same. The solution set is four dimensional.

that is to say, the effective index of refraction in the discrete medium is $(b \arcsin(a))/(a \arcsin(b))$, so modified from 1 by $O(l_p)$.

Small inhomogeneities of a and b in the fabric of spacetime over large regions could yield interesting frequency shifts in the index of refraction, and thus, e.g., whistler waves over cosmic distances. It might be worth investigating application of the umbral deformation functional (34) on such waves, to access long range effects of microscopic qualifications of the type considered.

A further, more technically challenging application of the umbral tranforms proposed might attain significance on nonlinear, solitonic phenomena, such as, e.g., the one-soliton solution of the continuum Sine-Gordon equation,

$$(\partial_x^2 - \partial_t^2)f(x, t) = \sin(f). \quad (42)$$

The corresponding umbral deformation of the equation itself would now also involve a deformed potential $\sin(f(x \frac{2}{T_x+T_x^{-1}}, t \frac{2}{T_t+T_t^{-1}})) \cdot 1$ on the right-hand side, for the Δ_s deformation—and $\sin(f(xT_x^{-1}, tT_t^{-1})) \cdot 1$ for the Δ_+ deformation.

As illustrated here, rather than solving difficult nonlinear difference equations, one may instead infer the umbral transform that, e.g., the continuum one-soliton solution maps to,

$$F_s = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{\pi^2} \arctan \left(me^{\frac{\chi-v\tau}{\sqrt{1-v^2}}} \right) e^{-i\chi p - i\tau k} \left(ipb + \sqrt{1 - (pb)^2} \right)^{x/b} \left(ika + \sqrt{1 - (ka)^2} \right)^{t/a}. \quad (43)$$

For the Δ_+ deformation, one would have, instead,

$$F_+ = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{\pi^2} \arctan \left(me^{\frac{\chi-v\tau}{\sqrt{1-v^2}}} \right) e^{-i\chi p - i\tau k} \left(ipb + 1 \right)^{x/b} \left(ika + 1 \right)^{t/a}. \quad (44)$$

Closed form evaluation of this integral appears complicated, but it could be plotted numerically and the representation could yield qualitative asymptotic insights on the $O(l_{Planck})$ modifications of such umbral solitons.

Likewise, the analog integrals with the continuum KdV soliton $f(x, t) = \frac{v}{2} \text{sech}^2(\frac{\sqrt{v}}{2}(x - vt))$ as input,

$$F_s = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{8\pi^2} v \text{sech}^2 \left(\frac{\sqrt{v}}{2}(\chi - v\tau) \right) e^{-i\chi p - i\tau k} \left(ipb + \sqrt{1 - (pb)^2} \right)^{x/b} \left(ika + \sqrt{1 - (ka)^2} \right)^{t/a}, \quad (45)$$

and, for the Δ_+ deformation,

$$F_+ = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{8\pi^2} v \text{sech}^2 \left(\frac{\sqrt{v}}{2}(\chi - v\tau) \right) e^{-i\chi p - i\tau k} \left(ipb + 1 \right)^{x/b} \left(ika + 1 \right)^{t/a}, \quad (46)$$

could be plotted numerically and compared to the Lax pair integrability machinery of Ref [5], or the results on a variety of discrete KdVs in ref [11]. These questions are left for a forthcoming study. The ingenuity of further particular applications of the deforming functionals proposed here is left to the reader.

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